Lecture: 3-7 Rates of Change in the Natural and Social Sciences

Example 1: A particle moves according to the law of motion $s = f(t) = t^4 - 4t + 1$. m + sec.

(a) Find the velocity at time t. What is the velocity after 2 seconds?

$$v(t) = s' = f'(t) = 4t^3 - 4$$

$$v(2) = 4(2^3) - 4 = 4(2^3 - 1) = \boxed{28 \text{ m/sec}}$$

(b) When is the particle at rest?

when $v(t) = 0$

$$0 = 4(t^3 - 1)$$

$$0 = t^3 - 1$$

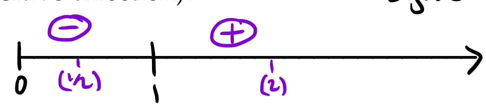
$$t^3 = 1$$

$$\boxed{t = 1 \text{ sec}}$$

(c) When is the particle moving forward (in the positive direction)?

sign of $v(t)$

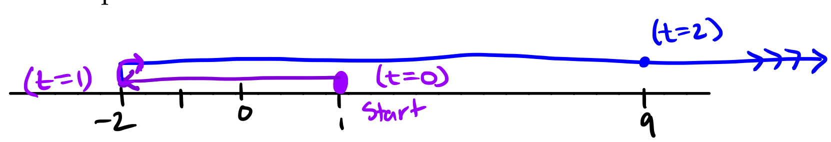
sign chart, test $v(t)$:



$\boxed{\text{on } [1, \infty) \text{ as } v(t) \text{ is postive on this interval}}$

(d) Draw a diagram to illustrate the motion of the particle.

To do:
① plot start
② plot at turnpoint(s)
③ final direction



① at $t=0$ position is $s(0) = 1$
② at $t=1$ position is $s(1) = 1 - 4 + 1 = -2$
③ at $t=2$ position is $s(2) = 16 - 8 + 1 = 9$

(e) Find the total distance traveled in the first 3 seconds.

0-1 second travels 3 units!

at $t=3$ position is $s(3) = 3^4 - 4(3) + 1 = 81 - 12 + 1 = 70$

from 1-3 seconds travels 2+70 units

Total is $\boxed{75}$ m

Example 2: If a ball is thrown vertically upward with a velocity of 80 ft/s, then its height after t seconds is $s = 80t - 16t^2$.

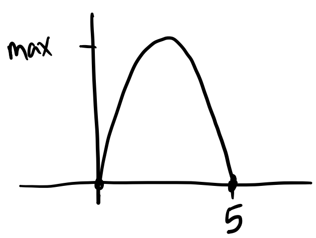
(a) What is the velocity of the ball after 2 seconds?

$v(t) = s'(t) = 80 - 32t$

$v(2) = 80 - 64$
$= \boxed{16 \text{ ft/sec}}$

(b) Sketch a rough graph of the ball's height as a function of time. Using Calculus, find the maximum height reached by the ball.

$s = 16t(5-t)$



max occurs where $v(t) = 0$

$80 - 32t = 0$

$80 = 32t$

$t = 80/32 = 10/4 = \boxed{5/2 \text{ sec}}$ when

How high: $s(5/2) = 80(5/2) - 16(5/2)^2$
$= 200 - 16(25/4)$
$= 200 - 100$
$= \boxed{100 \text{ ft}}$

Example 3: If a tank holds 1000 gallons of water, which drains from the bottom of the tank in 20 minutes, then Torricelli's Law gives the volume of water V remaining in the tank after 20 minutes as

$$V = 1000\left(1 - \frac{1}{20}t\right)^2 \quad 0 \le t \le 20.$$

Find the rate at which water is draining from the tank after (a) 5 minutes, (b) 10 minutes and (c) 20 minutes. At what time is the water flowing out the fastest? Slowest?

The rate of change of V is $\frac{dV}{dt} = 1000(2)(1 - t/20)(-1/20)$
$= 2000(1 - t/20)(-1/20)$
$= -100(1 - t/20)$

a) $t = 5 \Rightarrow \frac{dV}{dt} = -100(1 - 1/4) = -100(3/4) = \boxed{-75 \text{ gal/min}}$

b) $t = 10 \Rightarrow \frac{dV}{dt} = -100(1 - 1/2) = -100(1/2) = \boxed{-50 \text{ gal/min}}$

c) $t = 20 \Rightarrow \frac{dV}{dt} = -100(1 - 1) = \boxed{0 \text{ gal/min}}$

Leaks out fastest in the beginning, slows down as time passes and is zero after 20 min when tank is empty

Example 4: The volume of a growing spherical cell is $V = \frac{4}{3}\pi r^3$, where the radius r is measured in micrometers. Find the average rate of change of V with respect to t when r changes from:

(a) 5 to 6 micrometers

$$\frac{V(6)-V(5)}{6-5} = \frac{4}{3}\pi \cdot 6^3 - \frac{4}{3}\pi\, 5^3$$

$$= \frac{4}{3}\pi\left(6^3-5^3\right)$$

$$= \frac{364\pi}{3}$$

$$\approx \boxed{381.180\ \mu m^2} \left(\frac{\mu m^3}{\mu m}\right)$$

(b) 5 to 5.1 micrometers

$$\frac{V(5.1)-V(5)}{5.1-5} = \frac{\left(\frac{4}{3}\pi \cdot (5.1)^3 - \frac{4}{3}\pi\, 5^3\right)}{0.1}$$

$$= \frac{4}{3}\pi\left(5.1^3 - 5^3\right) \div (0.1)$$

$$\approx \boxed{320.484\ \mu m^2}$$

(c) Find the instantaneous rate of change of V with respect to r when $r = 5$ micrometers.

$$\frac{dV}{dr} = \frac{4}{3}\pi \cdot 3r^2$$

$$\boxed{\frac{dV}{dr} = 4\pi r^2}$$

when $r = 5$,

$$\frac{dV}{dr} = 4\pi\left(5^2\right)$$

$$= 100\pi\ \mu m^3/\mu m$$

$$= \boxed{100\pi\ \mu m^2}$$

(d) Show that the rate of change of the volume of a sphere with respect to its radius is equal to its surface area. Why might this be true?

$\frac{dV}{dr} = 4\pi r^2$ and this is SA of sphere.

Think – if you take a sphere, say a spherical apple or spherical cherry and you increase the radius of said sphere (apple / cherry) by adding caramel or chocolate, you add about a surface area worth of volume.

Economics

Marginal Cost Function

- $C(x)$ is total cost for making x units of a commodity
- If we increase items made from x_1 to x_2
  $\Delta C = C(x_2) - C(x_1)$ is the change in cost
- avg rate of cost is $\dfrac{C(x_2)-C(x_1)}{x_2 - x_1}$ as $\Delta x \to 0$

goes to $C'(x)$, the instantaneous rate of change of cost

Example 5: The cost, in dollars, of producing x yards of a certain fabric is

$$C(x) = 1200 + 12x - 0.1x^2 + 0.0005x^3$$

(a) Find the marginal cost function.

$$\boxed{C'(x) = 12 - 0.2x + 0.0015x^2}$$

(b) Find $C'(200)$ and explain its meaning. What does is predict?

$$C'(200) = 12 - 0.2(200) + 0.0015(200^2)$$
$$= 12 - \frac{2}{10}(200) + \frac{15}{10{,}000}\ 40{,}000$$
$$= 12 - 40 + 60$$
$$= \boxed{32\ \$/\text{yard}}$$

costs are increasing at a rate of \$32/yd when you produce 200yds

(c) Compare $C'(200)$ with the cost of manufacturing the 201st yard of fabric.

$$\Delta C = C(201) - C(200)$$
$$= \left(1200 + 12(201) - 0.1(201)^2 + 0.0005(201)^3\right)$$
$$- \left(1200 + 12(200) - 0.1(200)^2 + 0.0005(200)^3\right)$$
$$= \boxed{\$32.2005}$$